Journal of Algebra **226,** 1–14 (2000)
doi:10.1006/jabr.1999.8091, available online at http://www.idealibrary.com on IDEAL®

On the Ideal of an Embedded Join

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Communicated by David Buchsbaum

Received July 8, 1998

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INTRODUCTION

This work deals with the embedded join XY of two subschemes X, Y of $\mathbb{P}_k^n$. The join is again a subscheme of $\mathbb{P}_k^n$, which as a set consists of the closure of the union of all lines $\overline{xy}$ through distinct points x, y of X, Y (at least if k is algebraically closed). In the case $X = Y$ the join construction yields the classical secant variety.

Various authors have taken up the subject ([1, 4–6], for instance), often emphasizing the dimension of the join and the relation to intersection the-

* Partially supported by CNPq, Brazil.
† Partially supported by the National Science Foundation (DMS-9623259 and INT-9313693).

0021-8693/00 $35.00

ory. In the present article instead we aim at a more refined study of the (not necessarily reduced) ideal defining the embedding $XY \subset \mathbb{P}^n_k$.

One of the main tools we use is the deformation to the monomial case. We describe the precise behavior of the ideal of an embedded join under this mechanism (Section 2). The punch line here is given by Theorems 2.2 and 2.3. This reduction procedure leads us naturally to exploration of the join of subschemes defined by (not necessarily square-free) monomials (Section 3). As an application we estimate the initial degree of the ideals defining embedded joins and higher secant varieties, even in the non-monomial case (Section 4).

We illustrate our results with a few classical examples of varieties of determinantal type. The list includes generic matrices and generic symmetric and alternating matrices. Certain generalized catalecticant loci are also treated (Section 5).

1. PRELIMINARIES

Let k be a noetherian ring and let $R = k[\mathbf{X}]$ be a polynomial ring in n variables over k. Let $I, J \subset R$ be homogeneous ideals in the standard gradation of R. We introduce the main object of this work.

DEFINITION 1.1. The *join algebra* $\mathbb{J}(I,J)$ of I, J is the k-subalgebra $k[\mathbf{x} \otimes_k 1 - 1 \otimes_k \mathbf{y}] \subset R/I \otimes_k R/J$, where x_i (resp. y_i) denotes the image of X_i in R/I (resp. in R/J) and $\mathbf{x} \otimes 1 - 1 \otimes \mathbf{y} := \{x_1 \otimes 1 - 1 \otimes y_1, \ldots, x_n \otimes 1 - 1 \otimes y_n\}$.

In other words, $\mathbb{J}(I,J) = k[\mathbf{X} \otimes_k 1 - 1 \otimes_k \mathbf{X}]/\mathscr{D}(I,J)$, where $\mathscr{D}(I,J)$ is the contraction of the ideal $(I \otimes_k 1, 1 \otimes_k J)$.

To get an ideal sitting naturally in $k[\mathbf{X}]$, one proceeds further as follows. Consider the k-algebra homomorphisms $\pi_1: k[\mathbf{X} \otimes_k 1, 1 \otimes_k \mathbf{X}] \to k[\mathbf{X} \otimes_k 1]$ such that $\pi_1(X_i \otimes_k 1) = X_i \otimes_k 1$ and $\pi_1(1 \otimes_k X_i) = 0$, for all i, and $\pi_2: k[\mathbf{X} \otimes_k 1, 1 \otimes_k \mathbf{X}] \to k[1 \otimes_k \mathbf{X}]$ such that $\pi_2(X_i \otimes_k 1) = 0$ and $\pi_2(1 \otimes_k X_i) = 1 \otimes_k X_i$, for all i. Since the subring $k[\mathbf{X} \otimes_k 1 - 1 \otimes_k \mathbf{X}]$ is isomorphically mapped to its image under either π_1 or π_2, the same is true of the ideal $\mathscr{D}(I,J)$. Moreover, since $\mathscr{D}(I,J) \subset (I \otimes_k 1, 1 \otimes_k J)$, it is clear that $\pi_1(\mathscr{D}(I,J)) \subset I \otimes_k 1$ and $\pi_2(\mathscr{D}(I,J)) \subset 1 \otimes_k J$. Next, consider the isomorphisms $\tau_1: k[\mathbf{X} \otimes_k 1] \simeq k[\mathbf{X}]$ and $\tau_2: k[1 \otimes_k \mathbf{X}] \simeq k[\mathbf{X}]$ such that $\tau_1(X_i \otimes_k 1) = X_i$ and $\tau_2(1 \otimes_k X_i) = X_i$, respectively. Clearly, $I \otimes_k 1$ (resp. $1 \otimes_k J$) is isomorphically mapped to I (resp. J). Since $\mathscr{D}(I,J) \subset k[\mathbf{X} \otimes_k 1 - 1 \otimes_k \mathbf{X}]$, the images by $\tau_1 \circ \pi_1$ and by $\tau_2 \circ \pi_2$ of a homogeneous element of $\mathscr{D}(I,J)$ coincide in $k[\mathbf{X}]$ (up to a sign). Therefore, $\mathscr{D}(I,J)$ is isomorphic to its common image in $k[\mathbf{X}]$, and it is contained in both I and J. This common image is the (*embedded*) *join ideal* of I, J; it will be denoted $\mathfrak{J}(I,J)$. Thus, with this notation, $\mathfrak{J}(I, J) \subset I \cap J$, and the join algebra is isomorphic to $R/\mathfrak{J}(I, J)$. If $I = J$ we will refer to the join algebra and the join ideal as the *secant algebra* and the *secant ideal*, respectively.

Note that $\dim \mathbb{J}(I, J) \le \min\{n, \dim R/I + \dim R/J\}$ in the case where k is a field, as follows immediately from the definitions. If equality is attained one says that the join algebra has the *expected* (or *maximal*) *dimension*.

The following proposition sums up a few of the most elementary properties of the join construction.

PROPOSITION 1.2. *Let k be a field and let $I, J \subset R = k[\mathbf{X}]$ be homogeneous ideals.*

(i) *If $I = I_1 \cap I_2$ is a decomposition of I into homogeneous ideals then $\mathfrak{J}(I, J) = \mathfrak{J}(I_1, J) \cap \mathfrak{J}(I_2, J)$.*

(ii) *If k is perfect then $\sqrt{\mathfrak{J}(I, J)} = \mathfrak{J}(\sqrt{I}, \sqrt{J})$; in particular, if I and J are radical ideals then so is $\mathfrak{J}(I, J)$.*

(iii) *If I and J are geometrically prime ideals then $\mathfrak{J}(I, J)$ is a prime ideal.*

(iv) *If I and J are primary ideals whose respective radicals are geometrically prime then $\mathfrak{J}(I, J)$ is a primary ideal.*

Proof. (i) As $R \otimes_k R/J$ is a flat module over R (acting on the first factor), it follows that $((I_1 \cap I_2) \otimes_k 1, 1 \otimes_k J) = (I_1 \otimes_k 1, 1 \otimes_k J) \cap (I_2 \otimes_k 1, 1 \otimes_k J)$. Since intersection commutes with contraction of ideals, one obtains $\mathcal{D}(I_1 \cap I_2, J) = \mathcal{D}(I_1, J) \cap \mathcal{D}(I_2, J)$ for the non-embedded join ideals. Hence the same holds for the corresponding join ideals.

(ii) Since k is a perfect field, $(\sqrt{I} \otimes_k 1, 1 \otimes_k \sqrt{J})$ is a radical ideal. Therefore, one obtains $\sqrt{(I \otimes_k 1, 1 \otimes_k J)} = (\sqrt{I} \otimes_k 1, 1 \otimes_k \sqrt{J})$. Finally, contracting to the diagonal subalgebra $k[\mathbf{X} \otimes_k 1 - 1 \otimes_k \mathbf{X}]$, one deduces the assertion.

(iii) Let $\bar{k}$ be the algebraic closure of k. Setting $\bar{R} = R \otimes_k \bar{k} = \bar{k}[\mathbf{X}]$, then $I\bar{R}$, $J\bar{R}$ are prime ideals by hypothesis, hence $\bar{R}/I\bar{R} \otimes_{\bar{k}} \bar{R}/J\bar{R}$ is a domain. Therefore, its subring $R/I \otimes_k R/J$ is a domain, too. It follows that the join algebra $\mathbb{J}(I, J)$ is a domain.

(iv) Say, $\operatorname{Ass} R/I = \{P\}, \operatorname{Ass} R/J = \{Q\}$. Since $R/I \to R/I \otimes_k R/J$ is flat, $\operatorname{Ass}(R/I \otimes_k R/J) = \operatorname{Ass}(R/P \otimes_k R/J)$ ([11, 23.2]). By the same token, $\operatorname{Ass}(R/P \otimes_k R/J) = \operatorname{Ass}(R/P \otimes_k R/Q)$. On the other hand, since P and Q are geometrically prime, it follows by part (iii) that $\operatorname{Ass}(R/P \otimes_k R/Q)$ consists of one single element. Thus, $R/I \otimes_k R/J$ has only one associated prime, and the same holds for the subring $k[\mathbf{x} \otimes_k 1 - 1 \otimes_k \mathbf{y}]$. ∎

2. JOINS UNDER DEFORMATION

Here we briefly recall the Bayer deformation mechanism [2] in the form that suits our purpose. For that we follow closely the exposé of [3, Sect. 15.8].

Throughout, k is assumed to be a field. Let $R = k[\mathbf{X}] = k[X_1, \ldots, X_n]$ and let $>$ stand for a monomial term order on R. A weight function $\lambda: \mathbb{Z}^n \to \mathbb{Z}$ induces a partial order $>_\lambda$ on the set of monomials of R, to wit, $\mathbf{X}^\alpha >_\lambda \mathbf{X}^\beta \Leftrightarrow \lambda(\alpha) > \lambda(\beta)$ (in the ordinary order of $\mathbb{Z}$). Let λ be one such function. Given $g = \sum c_\alpha \mathbf{X}^\alpha \in R$, let $\mu(g) = \max_{c_\alpha \neq 0}\{\lambda(\alpha)\}$; set $\operatorname{in}_{>_\lambda}(g) = \sum_{\lambda(\alpha)=\mu(g)} c_\alpha \mathbf{X}^\alpha$ and $\tilde{g} = t^{\mu(g)} g(t^{-\lambda(e_1)}X_1, \ldots, t^{-\lambda(e_n)}X_n)$, where t is an indeterminate over R and $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{Z}^n$. Furthermore, for an ideal $I \subset R$, we set

$$\operatorname{in}_{>_\lambda}(I) := (\{\operatorname{in}_{>_\lambda}(g) \mid g \in I\}) \qquad \text{and} \qquad \tilde{I} := (\{\tilde{g} \mid g \in I\}) \subset R[t].$$

According to [3, Sect. 15.8], the basic result of Bayer can be summarized as follows.

PROPOSITION 2.1. *Given a term order $>$ on R and a finite collection $\{I_1, \ldots, I_s\}$ of ideals, there exists a weight function $\lambda: \mathbb{Z}^n \to \mathbb{Z}$ such that, if $>_\lambda$ denotes the induced partial order on the monomials of R, then $\operatorname{in}_{>_\lambda}(I_j) = \operatorname{in}_{>}(I_j)$, $j = 1, \ldots, s$. Moreover, for any such λ, one has*

(i) *$R[t]/\tilde{I}_j$ is a free $k[t]$-module, for $j = 1, \ldots, s$.*

(ii) *$(R[t]/\tilde{I}_j) \otimes_{k[t]} k[t, t^{-1}] \simeq (R/I_j) \otimes_k k[t, t^{-1}]$, for $j = 1, \ldots, s$.*

(iii) *$(R[t]/\tilde{I}_j) \otimes_{k[t]} (k[t]/(t)) \simeq R/\operatorname{in}_{>\lambda}(I_j)(=R/\operatorname{in}_{>_\lambda}(I_j))$, for $j = 1, \ldots, s$.*

We also need a somewhat more precise knowledge of the items of the proposition. The isomorphism in (iii) is induced by sending $t \mapsto 0$, while the one in (ii) is induced by the automorphism of $R[t, t^{-1}]$ that sends $X_i \mapsto t^{\lambda(e_i)}X_i$, $t \mapsto t$. Incidentally, note that by the latter, $\tilde{g} \mapsto t^{\mu(g)}g$, hence $\tilde{I}_j R[t, t^{-1}]$ maps isomorphically onto $I_j R[t, t^{-1}]$, for $j = 1, \ldots, s$. Furthermore, by (i), t is a non-zero divisor on $R[t]/\tilde{I}_j$.

From now on, we set $\operatorname{in}(I_j) = \operatorname{in}_>(I_j) = \operatorname{in}_{>_\lambda}(I_j)$, both the term order and the partial order being understood.

The main result of this part is the following "exchange" device.

THEOREM 2.2. *Let k be a field and let $I, J \subset R = k[\mathbf{X}]$ be homogeneous ideals. Then $\mathfrak{J}(I, J)^\sim = \mathfrak{J}(\tilde{I}, \tilde{J})$.*

Proof. We deform the three homogeneous ideals $I, J, \mathfrak{J}(I, J) \subset R$ simultaneously, using Proposition 2.1, thus obtaining the ideals $\tilde{I}$, $\tilde{J}$, $\mathfrak{J}(I, J)^\sim \subset R[t]$, respectively. Now, these ideals are homogeneous in the standard grading of $R[t] = k[t][\mathbf{X}]$ that has $R[t]_0 = k[t]$ (cf. the definitions before Proposition 2.1). In particular, the join ideal $\mathfrak{J}(\tilde{I}, \tilde{J}) \subset R[t]$ is well defined. Furthermore, the extensions of the "tilde" ideals to the ring $R[t, t^{-1}] = k[t, t^{-1}][\mathbf{X}]$ are still homogeneous in its standard gradation that has $R[t, t^{-1}]_0 = k[t, t^{-1}]$. Finally, it is clear that the extensions of the original ideals $I, J, \subset R$ to $R[t, t^{-1}]$ are homogeneous.

Using the structural isomorphism of Proposition 2.1(ii) at various levels, one obtains

$$\begin{aligned}
\mathfrak{J}(I, J)^{\sim} R[t, t^{-1}] &\simeq \mathfrak{J}(I, J) R[t, t^{-1}] && (1)\\
&\simeq \mathfrak{J}(IR[t, t^{-1}], JR[t, t^{-1}]) && (2)\\
&\simeq \mathfrak{J}(\tilde{I}R[t, t^{-1}], \tilde{J}R[t, t^{-1}]) && (3)\\
&\simeq \mathfrak{J}(\tilde{I}, \tilde{J}) R[t, t^{-1}], && (4)
\end{aligned}$$

where the isomorphisms in (1) and (3) go in opposite directions, the one in (2) results from flatness, and (4) is just localization at $\{t^s, s \geq 0\}$. Therefore, the final composite is induced by the identity map on $R[t, t^{-1}]$. In other words, $\mathfrak{J}(I, J)^{\sim}$ and $\mathfrak{J}(\tilde{I}, \tilde{J})$ are equal after localizing at $\{t^s, s \geq 0\}$. Thus, it suffices to show that t is a non-zero-divisor modulo both $\mathfrak{J}(I, J)^{\sim}$ and $\mathfrak{J}(\tilde{I}, \tilde{J})$. For the first, it follows from Proposition 2.1(i) (cf. also the comments after that proposition). By the same token, $(R[t]/\tilde{I}) \otimes_{k[t]} (R[t]/\tilde{J})$ is a free $k[t]$-module. Therefore, t is a non-zero-divisor modulo $(\tilde{I} \otimes_{k[t]} 1, 1 \otimes_{k[t]} \tilde{J})$, hence also modulo the contraction $\mathfrak{D}(\tilde{I}, \tilde{J})$ of this ideal. It follows that t is a non-zero-divisor modulo $\mathfrak{J}(\tilde{I}, \tilde{J})$, as required. ∎

THEOREM 2.3. *With the assumptions as above, one has* $\mathrm{in}(\mathfrak{J}(I, J)) \subset \mathfrak{J}(\mathrm{in}(I), \mathrm{in}(J))$.

Proof. One has a commutative diagram of homomorphisms of rings,

$$\begin{array}{ccc}
(R[t]/\tilde{I}) \otimes_{k[t]} (R[t]/\tilde{J}) & \xrightarrow{\mathrm{mod}(t)} & (R/\mathrm{in}(I)) \otimes_k (R/\mathrm{in}(J)) \\
\cup & & \cup \\
R[t]/\mathfrak{J}(\tilde{I}, \tilde{J}) & \xrightarrow{\mathrm{kills}(t)} & R/\mathfrak{J}(\mathrm{in}(I), \mathrm{in}(J)) \\
\| & & \varphi \uparrow \\
R[t]/\mathfrak{J}(I, J)^{\sim} & \xrightarrow{\mathrm{mod}(t)} & R/\mathrm{in}(\mathfrak{J}(I, J)),
\end{array}$$

where the top vertical inclusions follow from the identification of the homogeneous coordinate ring of the join with the join algebra and the lower leftmost vertical map comes from Theorem 2.2. This diagram then forces the existence of the rightmost lower vertical map φ, and the latter is induced by the identity on R. Therefore, $\mathrm{in}(\mathfrak{J}(I,J)) \subset \mathfrak{J}(\mathrm{in}(I),\mathrm{in}(J))$, as claimed. ∎

One gets equality $\mathrm{in}(\mathfrak{J}(I,J)) = \mathfrak{J}(\mathrm{in}(I),\mathrm{in}(J))$ if and only if the middle horizontal map in the preceding diagram is also reduction modulo (t). The following example shows that equality fails to hold in general.

EXAMPLE 2.4. [12, Example 5.8]. Let $I \subset R = k[X_1,\dots,X_6]$ be the ideal generated by the following polynomials:

$$f_1 = X_1X_2 - X_2X_5 + X_3X_5 - X_5X_6,$$
$$f_2 = X_1X_3 + X_1X_4 + X_1X_6 + X_4X_5,$$
$$f_3 = X_1X_6 + X_3X_5, \qquad f_4 = X_2X_3 + X_2X_6 - X_3X_6 + X_6^2,$$
$$f_5 = X_2X_4 + X_3X_6, \qquad f_6 = X_3^2 + X_3X_4 + X_3X_6 - X_4X_6.$$

Then I has the following features:

- R/I is an isolated singularity.
- The generators form a Gröbner basis in the lexicographic order with $X_1 > X_2 > X_4 > X_5 > X_3 > X_6$.
- In the above term order, $\mathrm{in}(I) = (X_1X_2,\ X_1X_4,\ X_1X_6,\ X_2X_3,\ X_2X_4,\ X_3X_4)$ is the edge ideal of a graph with exactly two odd cycles, and they have order 3.

As it turns out, $\mathfrak{J}(I,I) = 0$, while $\mathfrak{J}(\mathrm{in}(I),\ \mathrm{in}(I)) = (X_1X_2X_4,\ X_2X_3X_4)$ (see also Proposition 5.1). We note, incidentally, that R/I is Cohen–Macaulay, therefore $\mathrm{Proj}(R/I)$ embeds as a non-singular arithmetically Cohen–Macaulay surface in $\mathbb{P}^5$, whose secant variety is the whole of $\mathbb{P}^5$. As for $R/\mathrm{in}(I)$, $\mathrm{Proj}(R/\mathrm{in}(I))$ embeds as an arithmetically Cohen–Macaulay surface whose secant variety is non-equidimensional of dimension 4.

COROLLARY 2.5. *With the same notation as above, one has* $\dim \mathbb{J}(\mathrm{in}(I),\ \mathrm{in}(J)) \le \dim \mathbb{J}(I,J)$. *In particular, if* $\mathbb{J}(\mathrm{in}(I),\ \mathrm{in}(J))$ *has the expected dimension then so does* $\mathbb{J}(I,J)$.

Proof. This follows from Theorem 2.3 and the fact that the dimension does not change when passing to initial ideals (by Hilbert function theory). ∎

REMARK 2.6. One can go one step further, by taking radicals of the initial ideals. From the definition of the join algebra it is immediate that $\dim \mathbb{J}(\sqrt{\mathrm{in}(I)},\ \sqrt{\mathrm{in}(J)}) = \dim \mathbb{J}(\mathrm{in}(I),\ \mathrm{in}(J))$. This says that a sufficient condition for $\mathbb{J}(I,J)$ to have the expected dimension is that $\mathbb{J}(\sqrt{\mathrm{in}(I)}, \sqrt{\mathrm{in}(J)})$ does. This takes us to the study of the square-free monomial case.

3. JOINS OF MONOMIAL IDEALS

We keep the notation of the previous section. By definition, we have a k-algebra surjection $\mathbb{J}(I, J) \simeq R/\mathfrak{J}(I, J) \twoheadrightarrow R/I \cap J$, and we seek to determine the embedded join ideal naturally as a subideal of $I \cap J$. In this section, we will explicitly determine the embedded join ideal in the case where both I and J are generated by (not necessarily square-free) monomials.

PROPOSITION 3.1. *Let $I, J \subset R$ be ideals generated by monomials.*

(i) *If* char $k = 0$ *then $\mathfrak{J}(I, J)$ is generated by the set of monomials F in $\mathbf{X}$ such that any (necessarily monomial) factorization $F = GH$ (allowing $G = 1$ or $H = 1$) implies that either $G \in I$ or $H \in J$.*

(ii) *If* char $k = p > 0$ *then $\mathfrak{J}(I, J)$ is generated by the set of monomials F in $\mathbf{X}$ satisfying the following property: if $F = (X_1^{p^{e_1}})^{\alpha_1} \cdots (X_n^{p^{e_n}})^{\alpha_n}$ is the unique expression of F with p not dividing α_i for every i, then given any factorization $Z_1^{\alpha_1} \ldots Z_n^{\alpha_n} = G(\mathbf{Z})H(\mathbf{Z})$ (with $\mathbf{Z} = \{Z_1, \ldots, Z_n\}$ new variables), either $G(X_1^{p^{e_1}}, \ldots, X_n^{p^{e_n}}) \in I$ or $H(X_1^{p^{e_1}}, \ldots, X_n^{p^{e_n}}) \in J$.*

(iii) *(Arbitrary characteristic) If I and J are generated by square-free monomials in $\mathbf{X}$ then $\mathfrak{J}(I, J)$ is generated by the set of square-free monomials F in $\mathbf{X}$ such that any factorization $F = GH$ implies that either $G \in I$ or $H \in J$.*

Proof. We first prove that $\mathfrak{J}(I, J)$ is generated by monomials in all cases and that it is generated by square-free monomials if I and J are. It suffices to show this claim as well as assertions (i) and (ii) for the ideal $\mathcal{D}(I, J) \subset k[\mathbf{X} - \mathbf{Y}] = k[X_i - Y_i | 1 \le i \le n]$. So let an arbitrary $F \in k[\mathbf{X} - \mathbf{Y}]$ be written as a k-linear combination of monomials in the differences $X_i - Y_i$. Expanding to get a combination of monomials in $\mathbf{X}, \mathbf{Y}$, one clearly sees that no two such monomials cancel against each other. On the other hand, if $F \in (I(\mathbf{X}), J(\mathbf{Y}))$, since the latter is generated by monomials then each monomial of F (in $\mathbf{X}, \mathbf{Y}$) is a multiple of some monomial of $I(\mathbf{X})$ or of $J(\mathbf{Y})$. We can therefore reconstitute any original monomial of F (in the differences $X_i - Y_i$) as a k-linear combination of monomials from either $I(\mathbf{X})$ or $J(\mathbf{Y})$. Furthermore, it is clear in this argument that if $I(\mathbf{X})$ and $J(\mathbf{Y})$ are generated by square-free monomials then $\mathcal{D}(I, J)$ is generated by square-free monomials in the differences $X_i - Y_i$ (see also Proposition 1.2(ii)).

(i) By the above, it suffices to prove that a monomial F in the differences $X_i - Y_i$ is contained in $\mathcal{D}(I, J)$ if and only if $F(\mathbf{X} - \mathbf{Y}) = G(\mathbf{X} - \mathbf{Y})H(\mathbf{X} - \mathbf{Y})$ implies $G(\mathbf{X} - \mathbf{Y}) \in I(\mathbf{X} - \mathbf{Y})$ or $H(\mathbf{X} - \mathbf{Y}) \in J(\mathbf{X} - \mathbf{Y})$. So let $F = F(\mathbf{X} - \mathbf{Y})$ be such a monomial. The monomials in $\mathbf{X}, \mathbf{Y}$ that appear in F with non-zero coefficients are exactly the products $G(\mathbf{X})H(\mathbf{Y})$, where G,

H are monomials, with $F(\mathbf{X}-\mathbf{Y}) = G(\mathbf{X}-\mathbf{Y})H(\mathbf{X}-\mathbf{Y})$. Thus, $F \in \mathcal{D}(I, J)$ if and only if for every such G, H, one has $G(\mathbf{X}) \in I$ or $H(\mathbf{Y}) \in J(\mathbf{Y})$, which means that $G(\mathbf{X}-\mathbf{Y}) \in I(\mathbf{X}-\mathbf{Y})$ or $H(\mathbf{X}-\mathbf{Y}) \in J(\mathbf{X}-\mathbf{Y})$.

(ii) One proceeds as in the proof of part (i), noticing that now the monomials in $\mathbf{X}, \mathbf{Y}$ that appear in $F = ((X_1 - Y_1)^{p^{e_1}})^{\alpha_1} \cdots ((X_n - Y_n)^{p^{e_n}})^{\alpha_n}$ with nontrivial coefficients are exactly the monomials $G(X_1^{p^{e_1}}, \ldots, X_n^{p^{e_n}})$ $H(Y_1^{p^{e_1}}, \ldots, Y_n^{p^{e_n}})$, with $Z_1^{\alpha_1} \ldots Z_n^{\alpha_n} = G(\mathbf{Z})H(\mathbf{Z})$.

(iii) By the above, $\mathfrak{J}(I, J)$ is generated by square-free monomials. Furthermore, intersecting the monomial generating sets obtained in (i) or (ii), respectively, with the set of all square-free monomials yields exactly the set stated in this part. ∎

Parts (i) and (iii) of the above proposition can be paraphrased by saying that $\mathfrak{J}(I, J)$ is generated by those (square-free) monomials F, so that for any factorization $F = GH$ either one factor is in both ideals I and J, or both factors are in one ideal I or J. In the square-free case, one has another way of looking at the join ideal, which could also be deduced from the geometric interpretation:

PROPOSITION 3.2. *Let $I = P_1 \cap \cdots \cap P_r$ and $J = Q_1 \cap \cdots \cap Q_s$ be the prime decompositions of ideals $I, J \subset R = k[\mathbf{X}]$ generated by square-free monomials (hence the P_k's and the Q_l's are generated by variables of R). Then $\mathfrak{J}(I, J) = \bigcap_{k,l}[P_k, Q_l]$, where $[P_k, Q_l]$ denotes the ideal generated by the variables common to P_k and Q_l.*

Proof. By Proposition 1.2(i), we are reduced to the join $\mathfrak{J}(P, Q)$, where P, Q are (geometrically prime) ideals generated by variables. Now $\mathfrak{J}(P, Q)$ is prime by Proposition 1.2(iii), and then Proposition 3.1(iii) implies that $\mathfrak{J}(P, Q) = [P, Q]$. ∎

COROLLARY 3.3. *Let Δ, Δ' be simplicial complexes on the same set of vertices V and let I, J stand for the respective face ideals. Then $\mathbb{J}(I, J)$ is the Stanley–Reisner ring of the simplicial complex whose facets are the maximal elements of the set $\{F \cup F' \mid F, F' \text{ facets of } \Delta, \Delta'\}$. In particular, if $\Delta = \Delta'$ then $\mathfrak{J}(I, I) = 0$ if and only if there exist facets $F, F' \in \Delta$ such that $F \cup F' = V$.*

Proof. Recall that the minimal primes of a Stanley–Reisner ring are exactly the ideals generated by the variables corresponding to the complement of a facet of the corresponding simplicial complex. The assertion is thus a consequence of Proposition 3.2. ∎

The results of Proposition 3.2 lead to a formula for the dimension of $\mathbb{J}(I, J)$ if I and J are generated by (not necessarily square-free) monomials. In particular, one obtains a characterization of when the join algebra has the expected (i.e., maximal) dimension.

PROPOSITION 3.4. *Let $I, J \subset R = k[X_1, \dots, X_n]$ be ideals generated by monomials and let $P_1, \dots, P_r$ and $Q_1, \dots, Q_s$ be the minimal primes of I and J, respectively. Then*

(i) $\dim \mathbb{J}(I, J) = \max_{k,l}\{\dim R/P_k + \dim R/Q_l - \dim R/(P_k + Q_l)\}$.

(ii) $$\dim \mathbb{J}(I, J) = \begin{cases} n \Leftrightarrow \mathrm{ht}(P_k + Q_l) \\ \quad = \mathrm{ht}\, P_k + \mathrm{ht}\, Q_l & \text{for some } k, l \\ \dim R/I + \dim R/J \\ \quad \Leftrightarrow \mathrm{ht}(P_k + Q_l) = n & \text{for some } P_k \text{ and } Q_l \\ & \text{of minimal height.} \end{cases}$$

Proof. (i) We may replace I and J by $\sqrt{I}$ and $\sqrt{J}$, respectively. Thus, we assume that I, J are generated by square-free monomials. In this case, the formula follows from Proposition 3.2 since $\mathrm{ht}\, P_k + \mathrm{ht}\, Q_l - \mathrm{ht}(P_k + Q_l) = \mathrm{ht}[P_k, Q_l]$.

(ii) This is an immediate consequence of part (i). ∎

4. INITIAL DEGREE OF JOIN IDEALS

In this section we apply the above results to obtain lower bounds for the initial degree of $\mathfrak{J}(I, J)$, where I, J are not necessarily monomial ideals. In other words we find conditions for when the embedded join cannot lie on a hypersurface of low degree. Sometimes this even allows one to deduce that $\mathfrak{J}(I, J) = 0$.

PROPOSITION 4.1. *Let $I, J \subset k[\mathbf{X}] = k[X_1, \dots, X_n]$ be ideals generated by square-free monomials of degrees at least r and s, respectively. Then $\mathfrak{J}(I, J)$ is generated by square-free monomials of degrees at least $r + s - 1$. In particular, if $r + s > n + 1$ then $\mathfrak{J}(I, J) = 0$.*

Proof. The assertion is an immediate consequence of Proposition 3.1(iii). ∎

COROLLARY 4.2. *Let $I, J \subset k[\mathbf{X}] = k[X_1, \dots, X_n]$ be homogeneous ideals. If $\sqrt{\mathrm{in}(I)} \subset (\mathbf{X})^r$ and $\sqrt{\mathrm{in}(J)} \subset (\mathbf{X})^s$ and $r + s > n + 1$, then $\mathfrak{J}(I, J) = 0$.*

Proof. One applies Proposition 4.1 in conjunction with Corollary 2.5 and Remark 2.6. ∎

PROPOSITION 4.3. *Let $I, J \subset k[\mathbf{X}] = k[X_1, \dots, X_n]$ be homogeneous ideals. If $\mathrm{ht}\, I = g$ and $\sqrt{\mathrm{in}(J)} \subset (\mathbf{X})^{g+1}$, then $\mathfrak{J}(I, J) = 0$.*

Proof. By Corollary 2.5 and Remark 2.6 one may assume that I and J are generated by square-free monomials, in which case $\mathfrak{J}(I, J)$ has the same property by Proposition 3.1(iii). Thus if $\mathfrak{J}(I, J) \neq 0$ then certainly $X_1 \ldots X_n \in \mathfrak{J}(I, J)$. It follows that for any square-free monomial G of degree $n - g$ there exists a monomial H of degree g such that $GH = X_1 \ldots X_n \in \mathfrak{J}(I, J)$. As $H \notin J$ by hypothesis, Proposition 3.1(iii) implies that $G \in I$. Therefore, I contains all square-free monomials of degree $n - g$. But the latter generate an ideal of height $n - (n - g) + 1 = \operatorname{ht} I + 1$, which is absurd. ∎

THEOREM 4.4. *Let* char $k = 0$ *and let* $I, J \subset R = k[\mathbf{X}]$ *be homogeneous ideals. If* $I \subset (\mathbf{X})^r$ *and* $J \subset (\mathbf{X})^s$ *then* $\mathfrak{J}(I, J) \subset (\mathbf{X})^{r+s-1}$.

Proof. First notice that, quite generally, for a homogeneous ideal $K \subset R$ and a given term order, $K \subset (\mathbf{X})^t$ if and only if $\operatorname{in}(K) \subset (\mathbf{X})^t$. Thus, we may use Theorem 2.3 to reduce to the case where I, J are generated by monomials. But, in this case, the assertion follows from Proposition 3.1(i). ∎

This theorem says in particular that the embedded join of two nondegenerate subschemes of projective space cannot lie on a quadric hypersurface (at least if char $k = 0$).

Special cases of the join construction are the higher secant varieties: Let $I \subset R = k[X_1, \ldots, X_n]$ be a homogeneous ideal and let t be a positive integer. Define ideals $\mathfrak{J}_t(I) \subset R$ inductively by setting $\mathfrak{J}_1(I) = \mathfrak{J}(I, I)$ and $\mathfrak{J}_t(I) = \mathfrak{J}(I, \mathfrak{J}_{t-1}(I))$ if $t > 1$. The subscheme $\operatorname{Sec}_t(X) = V(\mathfrak{J}_t(I)) \subset \mathbb{P}^{n-1}$ is called the *variety of t-secants* of $X = V(I)$. As a set, $\operatorname{Sec}_t(X)$ is the closure of the union of all t-dimensional linear subspaces spanned by $t + 1$ points of X (at least if k is algebraically closed). Theorem 4.4 implies immediately:

COROLLARY 4.5. *Let* char $k = 0$ *and let* $I \subset k[\mathbf{X}]$ *be a homogeneous ideal. If* $I \subset (\mathbf{X})^r$ *then* $\mathfrak{J}_t(I) \subset (\mathbf{X})^{(t+1)(r-1)+1}$

Thus the variety of t-secants of a nondegenerate subscheme of projective space cannot be contained in a hypersurface of degree $t + 1$ (at least if char $k = 0$)—a result originally observed by M. Catalano-Johnson (a proof given by M. Catalisano appears in [7, Lecture 7]).

5. EXAMPLES

Edge Ideals

An ideal $I \subset k[\mathbf{X}]$ generated by square-free monomials of degree 2 is naturally associated with a simple graph G in which the vertices correspond to the variables $\mathbf{X}$ and the edges to the given generators of I. We then speak of I as the *edge ideal* of G and denote it by $I(G)$ to keep track of the corresponding graph. A *cycle* of G is a set of edges of G forming a polygon. The algebraic counterpart is given by a subset $\{X_{i_1}, \ldots, X_{i_r}\}$ of the variables with the property that $\{X_{i_1}X_{i_2}, X_{i_2}X_{i_3}, \ldots, X_{i_{r-1}}X_{i_r}, X_{i_r}X_{i_1}\}$ correspond to edges of G. The cycle is *even* (resp. *odd*) if r is even (resp. odd). The cycle is said to be *chordless* (in the graph) if no other pair involving these variables corresponds to an edge of G (in the combinatorial language this means that the induced subgraph corresponding to the given vertices is the cycle itself).

PROPOSITION 5.1. *Let $I = I(G) \subset k[\mathbf{X}]$ be the edge ideal of a simple graph G. Then the secant ideal $\mathfrak{J}(I, I) \subset k[\mathbf{X}]$ is minimally generated by the monomials $X_{i_1} \ldots X_{i_r}$ such that $\{X_{i_1}, \ldots, X_{i_r}\}$ correspond to the vertices of a chordless odd cycle of G.*

Proof. By Proposition 3.1(iii), $\mathfrak{J}(I, I)$ is generated by square-free monomials $X_{i_1} \cdots X_{i_r}$ such that any bipartition of the set $\{X_{i_1}, \ldots, X_{i_r}\}$ contains a pair X_{i_j}, X_{i_k} corresponding to an edge of G. This means that the induced subgraph with vertices $\{X_{i_1}, \ldots, X_{i_r}\}$ is not bipartite and hence contains an odd cycle of G. Thus, the embedded secant ideal is generated by the monomials whose support corresponds to an odd cycle. Finally, if $C \subset G$, is an odd cycle having a chord which is an edge of G, then one of the two subcycles determined by this chord, say C', is odd. Clearly, the monomial whose support is C is a multiple of the one whose support is C'. This procedure shows that the monomials whose support is a chordless odd cycle form a set of minimal generators of the embedded secant ideal. ∎

Catalecticant Ideals

DEFINITION 5.2. Let $r \geq 1$ be an integer. A generic *r-catalecticant matrix* of size $m \times n$ $(2 \leq m \leq n)$ is a matrix of the form

$$\mathbf{X} = \begin{pmatrix} X_1 & X_2 & \cdots & X_n \\ X_{1+r} & X_{2+r} & \cdots & X_{n+r} \\ X_{1+2r} & X_{2+2r} & \cdots & X_{n+2r} \\ \vdots & \vdots & & \vdots \\ X_{1+(m-1)r} & X_{2+(m-1)r} & \cdots & X_{n+(m-1)r} \end{pmatrix}.$$

Note that the case $r = 1$ gives a Hankel matrix.

PROPOSITION 5.3. *Let $\mathbf{X}$ be a generic r-catalecticant matrix of size $m \times n$ $(2 \leq m \leq n)$ and let $I = I_m(\mathbf{X}) \subset k[X_1, \ldots, X_{n+(m-1)r}]$. Suppose that $n - m - 1 \leq r$.*

(i) *If $m \geq 3$ then $\mathfrak{J}(I, I) = 0$.*

(ii) *If $m = 2$ and $n - 3 < r$ then $\Im(I, I) = 0$; if $m = 2$ and $n - 3 = r$ then $\Im(I, I)$ is the principal ideal generated by the determinant of the following $(n-3)$-catalecticant matrix of size 3×3:*

$$\begin{pmatrix} X_1 & X_2 & X_3 \\ X_{n-2} & X_{n-1} & X_n \\ X_{2n-5} & X_{2n-4} & X_{2n-3} \end{pmatrix}.$$

In particular, $\mathbb{J}(I, I)$ always has the expected dimension.

Proof. In the range $n - m - 1 \leq r$, if $>$ is the lexicographic term order on $X_1 > \cdots > X_{n+(m-1)r}$ then $\mathrm{in}(I)$ is generated by the monomials along the diagonals of the maximal minors of $\mathbf{X}$ (cf. [10, III.3.8]). It follows, in particular, that the ideals $(X_1, \ldots, X_{n-m+1})$ and $(X_{m+(m-1)r}, \ldots, X_{n+(m-1)r})$ are prime ideals containing $\mathrm{in}(I)$. The condition $n - m - 1 \leq r$ implies $n - m + 1 \leq r + 2$. Clearly, for $m \geq 3$, one has $r + 2 < (m-1)r + m$. Therefore, for $m \geq 3$, we have the strict inequality $n - m + 1 < m + (m-1)r$. But the latter means that the two prime ideals have no common variables. By Proposition 3.2, $\Im(\mathrm{in}(I), \mathrm{in}(I)) = 0$. Then, by Theorem 2.3, $\Im(I, I) = 0$. This proves (i).

To prove (ii), note that for $m = 2$, our hypothesis boils down to $n - 3 \leq r$. Now, for $n - 3 < r$ the above argument still applies and, consequently, $\Im(I, I) = 0$ in this case. So, assume that $n - 3 = r$. Here the given matrix has the following shape:

$$\begin{pmatrix} X_1 & X_2 & \ldots & X_{n-2} & X_{n-1} & X_n \\ X_{n-2} & X_{n-1} & \ldots & X_{2n-5} & X_{2n-4} & X_{2n-3} \end{pmatrix}.$$

Since $\mathrm{in}(I)$ is generated by the products $X_i X_j$ along the diagonals of the 2×2 minors, one sees that the corresponding graph has only one odd cycle, namely, the triangle with vertices corresponding to the variables X_1, X_{n-1}, X_{2n-3}. By Proposition 5.1, $\Im(\mathrm{in}(I), \mathrm{in}(I)) = (X_1 X_{n-1} X_{2n-3})$. Since $\mathrm{in}(\Im(I, I)) \subset (X_1 X_{n-1} X_{2n-3})$ by Theorem 2.3, then $\mathrm{ht}\, \Im(I, I) = \mathrm{ht}\, \mathrm{in}(\Im(I, I)) \leq 1$. On the other hand, the 2×2 minors of the 3×3 matrix in the statement belong to I; hence the determinant of the matrix is an element of $\Im(I, I)$, which is seen by substituting $X_i - Y_i$ for X_i and expanding the determinant (cf. also [13, the proof of 4.11]). Since the determinant is an irreducible polynomial, it must generate the join ideal of I. ∎

The next examples are sufficiently known by geometric arguments (cf., e.g., [8, p. 145]). Our purpose is to show that the present method of descending to the initial ideal also yields these results.

Generic Determinantal Ideals

PROPOSITION 5.4. *Let* $\mathbf{X}$ *be a generic* $m \times n$ *matrix* (*a generic* $m \times m$ *symmetric matrix*, *respectively*), *and let* $I = I_s(\mathbf{X}) \subset k[\mathbf{X}]$ *with* $1 \le s \le m \le n$ (*with* $1 \le s \le m$, *respectively*). *Then* $\mathfrak{J}(I, I) = I_{2s-1}(\mathbf{X})$.

Proof. We give the argument in the generic case, the symmetric case requiring but a slight modification. Consider the so-called diagonal term order $>$ on $\mathbf{X}$ (i.e., the lexicographic term order induced by the decreasing ordering of the variables along successive rows (top to bottom)). It is well known that a Gröbner basis of I for this term order is given by the $s \times s$ minors of the matrix. Clearly, $\text{in}(I)$ is generated by the square-free monomials obtained by reading the main diagonals of the $s \times s$ minors. In particular, the prime ideals $(X_{ij} \mid 1 \le i \le m-s+1,\ 1 \le j \le n-s+1)$ and $(X_{ij} \mid s \le i \le m,\ s \le j \le n)$ contain $\text{in}(I)$. By Proposition 3.2, $\mathfrak{J}(\text{in}(I), \text{in}(I))$ is contained in the ideal $(X_{ij} \mid s \le i \le m-s+1,\ s \le j \le n-s+1)$, and by Theorem 2.3, $\text{in}(\mathfrak{J}(I, I)) \subset \mathfrak{J}(\text{in}(I), \text{in}(I))$. Therefore, $\text{ht}\,\mathfrak{J}(I, I) \le \max\{0, (m-2s+2)(n-2s+2)\} = \text{ht}\, I_{2s-1}(\mathbf{X})$.

On the other hand, by [13, the proof of 4.11], the $(2s-1) \times (2s-1)$ minors yield elements of the ideal $\mathfrak{J}(I, I)$. Since the minors of x generate the prime ideal $I_{2s-1}(\mathbf{X})$, the equality $\mathfrak{J}(I, I) = I_{2s-1}(\mathbf{X})$ follows suit. ∎

Generic Pfaffian Ideals

PROPOSITION 5.5. *Let* $\mathbf{X}$ *be a generic* $m \times m$ *alternating matrix and let* $I = \text{Pf}_{2s}(\mathbf{X}) \subset k[\mathbf{X}]$, *with* $1 \le 2s \le m$, *be the ideal generated by the* $2s \times 2s$ *Pfaffians of* $\mathbf{X}$. *Then* $\mathfrak{J}(I, I) = \text{Pf}_{4s-2}(\mathbf{X})$.

Proof. We may assume at the outset that k is a perfect field. With the lexicographic term order induced by ordering the variables $X_{1,m} > X_{1,m-1} > \cdots > X_{1,2} > X_{2,m} > X_{2,m-1} > \cdots > X_{2,3} > \cdots > X_{m-1,m}$, the $2s \times 2s$ Pfaffians of $\mathbf{X}$ form a Gröbner basis of I (cf. [9, 5.1]). Then $\text{in}(I)$ is generated by square-free monomials and is contained in the prime ideals $(X_{i,j} | s \le i < j \le m-s+1)$ and $(X_{i,j} | 1 \le i < j-2s+2 \le m-2s+2)$. As before, one deduces that

$$\begin{aligned} \text{ht}\,\mathfrak{J}(I, I) &\le \text{ht}\,\mathfrak{J}(\text{in}(I), \text{in}(I)) \\ &\le \max\left\{0,\ \frac{(m-4s+4)(m-4s+3)}{2}\right\} = \text{ht}\,\text{Pf}_{4s-2}(\mathbf{X}). \end{aligned}$$

On the other hand, notice that $I_{2s-1}(\mathbf{X}) \subset I$. It follows by [13, the proof of 4.11] that $I_{4s-2}(\mathbf{X}) \subset \mathfrak{J}(I, I)$. Therefore, $\text{Pf}_{4s-2}(\mathbf{X}) \subset \sqrt{I_{4s-2}(\mathbf{X})} \subset \sqrt{\mathfrak{J}(I, I)} = \mathfrak{J}(I, I)$, the last equality following from Proposition 1.2(ii). Therefore, we have the required equality. ∎

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